Resource Procurement for Matching Market: A Nash-in-Nash Approach

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October 18, 2024

Abstract

This paper examines the procurement of resources for a subsequent matching market. We present a three-stage model: 1) bilateral bargaining between procurers and resource suppliers, 2) suppliers' decisions on resource provision, and 3) the realization of a stable matching outcome. We adopt the Nash-in-Nash approach as the solution concept for bargaining, revealing a unique equilibrium where procurers are unable to incentivize suppliers, resulting in minimal procurement. However, we demonstrate that by committing to an assignment rule that reverses the order of assignment, the government can increase the number of procured resources in equilibrium. Our findings emphasize the social benefits of integrating the allocation and procurement problems.

JEL Classification Codes: C71; C78; Q54; L14.
Keywords: Evacuation; Bargaining; Assignment problem; Potential game; Discrete convex analysis.

1 Introduction

Matching theory broadly tackles the allocation of limited resources and has recently expanded beyond theoretical analysis to practical applications in real-world distribution challenges and empirical research. However, policymakers frequently face markets where the issue extends beyond inefficient allocation to a fundamental shortage of resources. While matching theory guides optimal allocations from a fixed set of resources, it overlooks how such resources are produced, why their scarcity merits allocation considerations, and what strategies policymakers might employ to boost resource supply.

To illustrate, consider the case of disaster evacuation. The United Nations Office for Disaster Risk Reduction (UNDRR) has emphasized the importance of disaster risk reduction, including the

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securing of evacuation sites, in response to escalating damage from disasters.[1] This is especially critical for individuals who are particularly vulnerable during emergencies.[2] [3] However, a recent UNDRR survey shows that global preparedness remains inadequate. In fact, 84% of persons with disabilities reported not having a personal disaster preparedness plan.[4] This situation can be framed as a classical assignment problem Shapley and Shubik (1971), where evacuation sites are assumed to be a fixed set of resources. Prior to a disaster, governments appoint procurers to secure a sufficient number of evacuation sites; however, this procurement has often been inadequate. A similar problem is evident in other contemporary challenges, such as the influx of immigrants into European countries and the provision of childcare services.

Our model consists of three stages: a bargaining stage, a supply stage, and an assignment stage. In the bargaining stage, we model the resource procurement by the government as bilateral negotiations between government agents and resource suppliers, where government agents can set compensations to incentivize suppliers to provide resources. In the supply stage, resource suppliers decide how many resources to provide to the market given the compensations, anticipating the assignment results. In the assignment stage, a matching between the resource side and the demand side is realized using the concept of a *stable outcome* (Shapley and Shubik, 1971) as the equilibrium concept for the matching outcome.[5]

The difficulty in the bargaining stage lies in externalities: each supplier's decision about how many resources to supply depends on the bargaining outcomes of others. As a reasonable solution concept for this setting, we adopt the Nash-in-Nash solution concept of Horn and Wolinsky (1988).

Our first result concerns the equilibrium of the supply stage. We show that the supply-stage game possesses a potential function (Monderer and Shapley, 1996); hence, there exists at least one pure-strategy equilibrium in terms of the number of procured resources. Furthermore, we

[1] UNDRR leads global efforts in this area, with the Sendai Framework for Disaster Risk Reduction 2015–2030 as its guiding agenda. One of the framework's four primary objectives is enhancing disaster preparedness for effective response. The full framework can be accessed at this link.

[2] Recent studies show that vulnerable populations suffer disproportionately in disasters. For example, during the Great East Japan Earthquake, the mortality rate for vulnerable groups was approximately 2.7 times higher than the average rate among evacuees Aikawa (2013).

[3] A global analysis similarly highlights that "the mortality rate of people with disabilities in natural disasters can be up to four times higher than that of people without disabilities, due to a lack of inclusive planning, accessible information, early warning systems, transportation, and discriminatory attitudes within institutions and society" Stein and Stein (2022). For further details, see the United Nations report at `https://www.un.org/development/desa/disabilities/wp-content/uploads/sites/15/2020/03/Final-Disability-inclusive-disaster.pdf`.

[4] The full report is available at this link.

[5] It is widely recognized that a form of the welfare theorem applies in this context: the equilibrium matching is characterized as the solution to the social welfare maximization problem. Consequently, an alternative perspective on this assignment stage is that the government enforces the efficient matching outcome in the market.

demonstrate that the supply stage has a generically unique Nash equilibrium due to the M-concavity property of the potential function (Murota, 1998).

Our second result reveals that, in the bargaining stage, suppliers are not incentivized to provide resources under realistic parameter settings; in equilibrium, the government cannot induce the supply of resources by offering appropriate compensation.

As our final result, we show that committing to an appropriate assignment rule in the assignment stage can lead to a positive number of procured resources in equilibrium. We consider the *reverse assignment rule*, in which the government decides the assignment in the reverse order of a stable outcome. When adopting this reverse assignment rule, there are cases where the government can ensure that suppliers provide a positive number of resources in equilibrium. Although this rule is neither optimal nor realistic, our findings emphasize the importance of integrating the procurement of resources with their assignment.

1.1 Related Work

One relevant strand of literature considers conservation goods (Harstad, 2016; Harstad and Mideksa, 2017; Li, Ashlagi and Lo, 2023). In that evacuation slots provide societal benefits while remaining unused by hospitals, our situation faces a problem similar to the deforestation issues addressed in these papers. While we share their focus on contract problems, we further consider situations where the government can change how the conserved slots are used during evacuations; that is, the integration of assignment and procurement structures is our novel contribution. The literature on the hold-out problem, such as Kominers and Weyl (2012), also considers similar issues. Unlike these studies, our model allows the government to procure a partial set of goods from suppliers, leading to more complex inter-supplier relationships than the network structures considered in Sarkar (2017); Sarkar and Gupta (2023).

Our paper presents a new application of the Nash-in-Nash approach, as developed in Horn and Wolinsky (1988). The tractability of the solution broadens its range of applications, particularly in empirical industrial organization for bilateral contracting analysis under certain forms of externality; a recent review can be found in Lee, Whinston and Yurukoglu (2021). In addition to the model, our paper represents a new application of discrete convex analysis to economics; recent examples include auctions Lehmann, Lehmann and Nisan (2006), matching Murota and Yokoi (2015), and congestion games Fujishige et al. (2015). To the best of our knowledge, this is the first paper to use discrete convex analysis to examine a bargaining problem.

1.2 Organization of This Paper

In Section 2, we introduce the three stages of our model: the bargaining stage, the supply stage, and the assignment stage. We describe each stage in separate subsections, following this sequence. In Section 3, we analyze the game's equilibrium. Here, we reverse the order of the analysis: we first examine the supply decision and then analyze the bargaining problem. In Section 4, we consider the possibility of intervention to the assignment stage for inducing the supply of evacuation sites in equilibrium. Finally, in Section 5, we summarize the paper and discuss its implications.

2 Model

In order to have the clear image of the problem setting, hereafter, we consider the problem of evacuation and the procurement of the evacuation slots, which is described in Introduction.

The government, denoted by G, tries to procure the evacuation slots from hospitals for individuals requiring medical care, whom we term *refugees*. We denote the set of refugees by R and each refugee by r. Initially, there are no available evacuation slots for them. We denote the hospitals offering evacuation slots by $h \in H$ and, in the basic model, we consider two hospitals case, indexed by $h = 1$ and $h = 2$. We define $H_0 \equiv H \cup \{\phi\}$ where ϕ denotes the outside option, or unmacth in the matching market. We use $h = 0$ to represent this unmatch result. For each hospital, the government has different procurers, denoted by $p \in \{1, 2\}$. They are independent agents: the contract information is not shared between these procures[6]. In our case, this addresses the situation where different administrative departments are responsible for each region due to the vertical division of government. If necessary, you can regard a hospital as a single district.

We assume that the social surplus generated by a matching of $h \in H_0$ and $r \in R$, denoted by Φ_{hr}, is common knowledge[7]. We normalize the utility for unmatch to 0, i.e., $\Phi_{0r} = 0$. To avoid ties between refugees and evacuation sites, we assume that Φ_{hr} differs for every pair of $h \in H$ and $r \in R$. For instance, we can assume $\Phi_{hr} \sim F$ for some continuous distribution F.

In the bargaining stage, the two procurers initiate negotiations with their respective hospitals

[6] This structure corresponds to the *delegated agent model* in the vertical contracts literature, which serves as a microfoundation for the Nash-in-Nash solution (Lee, Whinston and Yurukoglu, 2021). When the delegated agent model is not applicable, we can instead rely on the different microfoundation proposed by Collard-Wexler, Gowrisankaran and Lee (2019).

[7] In practice, the government in advance conduct surveys to get the knowledge about Φ_{hr}. Several studies takes this assumption to evaluate the actual and counterfactual matching algorithm (Ahani et al., 2023). Without this knowledge, we can rely on the empirical matching literature to estimate the surplus generated by pairs from the observed matching outcomes (Agarwal, 2015; Galichon and Salanié, 2021).

to prepare the evacuation slots. Through the bargaining the following three objects are determined: (1) the compensation c_h to hospital h, (2) the number of procured refugee slots m_h, and (3) the lump-sum payment to the hospital l_h. In the subsequent supply stage, the hospitals then decide on the number of slots to supply based on the compensation agreed upon in the bargaining stage. At last, in the assignment stage, the refugees are assigned to the procured slots, which constitutes a stable outcome of Shapley and Shubik (1971).

2.1 Assignment Stage

In this section, we describe the the assignment stage. At the onset of this stage, each hospital has already determined the number of evacuation slots to be supplied. This number for hospital $h \in H$ is denoted by m_h. And we set $m_0 = \infty$. In our baseline model, the government does not intervene to this stage and the matching outcome naturally follow a stable outcome (Shapley and Shubik, 1971).

It is well known that this stable outcome is also computed as the solution of the social welfare maximization problem under the capacity constraints. We denote the set of assignment rules as $\mathcal{G}$, which is collection of maps from R to H_0. Let $S(m_1, m_2; \Gamma)$ represent the social welfare obtained under the rule $\Gamma \in \mathcal{G}$ for the set (m_1, m_2). The specific rule aiming to maximize social surplus, denoted by Γ_{max}, is obtained through the following program:

$$S(m_1, m_2; \Gamma_{max}) \equiv \begin{bmatrix} \max\limits_{\mu \geq 0} & \sum\limits_{h,r} \Phi_{hr}\mu_{hr} & \\ \text{s.t.} & \sum_r \mu_{hr} \leq m_h & (h \in \{0, 1, 2\}) \\ & \sum_h \mu_{hr} \leq 1 & (r \in R) \end{bmatrix} \tag{1}$$

Hereafter, for simplicity, we denote the set of refugees matched with hospital $h \in \{0, 1, 2\}$ as μ_h, and $S(m_1, m_2)$ represents the social welfare under the assignment rule Γ_{max}.

2.2 Supply Stage

In this section, we describe how hospital h determines the number of slots to supply to the assignment stage. We do not assume that procurers can enforce the number of slots to be procured at this stage. Instead, the set of procured evacuation slots must satisfy an equilibrium condition in the supply stage. This reflects the situation where procurers lack the authority to compel hospitals to leave slots unoccupied. This restriction serves as a constraint on the contract space, as will be discussed in the following section.

We introduce additional notations: M_h represents the maximum number of slots in hospital h, and b_h denotes the benefits derived from exploiting a slot. The payoff function for hospital h is defined as the sum of profits from utilizing vacant slots, the utility from societal contributions, and the lump-sum payment. For hospital 1, this is expressed as:

$$\pi_1(m_1, m_2) = (M_1 - m_1)\, b_1 + m_1 c_1 + \alpha_1 S\,(m_1, m_2) + l_1, \tag{2}$$

where $\alpha_1 \geq 0$ is a parameter indicating the importance hospital 1 places on contributing to refugee matching. Hospital 2 has an analogous payoff function.

Hospital 1 maximizes its payoff $\pi_1(m_1, m_2)$ by choosing m_1. Given a compensation c_1, we define the best response correspondence as $m_1^\star : \mathbb{N} \to 2^{\mathbb{N}}$, mapping the number of slots procured in hospital 2 to the set of optimal number of slots in hospital 1. For simplicity, we omit the dependency on c_1 in this notation. Similarly, we define $m_2^\star : \mathbb{N} \to 2^{\mathbb{N}}$ as the best response correspondence for hospital 2. We expect that a Nash equilibrium is achieved in the supply stage: (m_1^N, m_2^N) such that $m_1^N \in m_1^\star(m_2^N)$ and $m_2^N \in m_2^\star(m_1^N)$. It is important to note that the uniqueness of the Nash equilibrium in this game is not guaranteed, and the set of equilibrium outcomes depends on the compensation values (c_1, c_2).

2.3 Bargaining Stage

In this section, we define a bargaining problem between procurers and hospitals and its solution. There are two separate bargaining processes, one for each hospital, $h = 1$ and $h = 2$. For each bargaining, the parties discuss c_h, m_h, and l_h. The complexity of the situation arises from the interdependence of the two bargainings: m_h must belong to the set $m_h^\star(m_{-h})$ for each h. Additionally, the social welfare achieved, which directly influences the procurer's preferences, is determined by the pair (m_1, m_2). To address this, we apply the Nash-in-Nash solution as the solution concept (Horn and Wolinsky, 1988; Lee, Whinston and Yurukoglu, 2021). To define this solution, we first introduce the preferences of the procurers, the disagreement point between the procurers and hospitals, and the bargaining powers of them.

The procurer's payoff function is the net benefit of procurement: upon setting c_1 and c_2, and realizing (m_1, m_2) as the number of evacuation slots,

$$S(m_1, m_2) - c_h m_h - l_h.$$

For the disagreement point, we define the benefits achieved in the event of a breakdown in bargaining. For hospital 1, the utility at the disagreement point is the sum of the benefit from fully utilizing its capacity and the societal utility during the assignment stage:

$$M_1 b_1 + \alpha_1 S\left(0, m_2^\star(0)\right),$$

where $m_2^\star(0)$ represents hospital 2's best response to receiving no evacuation slots from hospital 1. At this disagreement point, the procurer's benefit is $S\left(0, m_2^\star(0)\right)$. Since the procurer cannot enforce the allocation of evacuation slots, the number of slots for hospital 2 must be the best response to $m_1 = 0$. We define an analogous disagreement point for the bargaining with hospital 2. The bargaining power of hospital h is denoted by $\beta_h \in [0, 1]$ where the bargaining power of the corresponding procurer is defined as $1 - \beta_h$.

Now we can consider the Nash bargaining problem between the procurer and hospital and it solution. For the case of hospital 1, given (c_2, m_2, l_2), *Nash bargaining solution* is the triplet (c_1^B, m_1^B, l_1^B) maximizing the following generalized Nash product:

$$\begin{aligned} c_1^B, m_1^B, l_1^B = \underset{c_1, m_1 \in m_1^\star(m_2), l_1}{\arg\max} \; & \left(S(m_1, m_2) - c_1 m_1 - l_1 - S(0, m_2^\star(0))\right)^{\beta_1} \\ & \times \left(\alpha_1 S(m_1, m_2) - (b_1 - c_1) m_1 + l_1 - \alpha_1 S(0, m_2^\star(0))\right)^{1-\beta_1}, \end{aligned} \tag{3}$$

We also have the analogous problem and solution for hospital 2's bargaining given (c_1, m_1, l_1)[8].

$$\begin{aligned} c_2^B, m_2^B, l_2^B = \underset{c_2, m_2 \in m_2^\star(m_1), l_2}{\arg\max} \; & \left(S(m_1, m_2) - c_2 m_2 - l_2 - S(m_1^\star(0), 0)\right)^{\beta_2} \\ & \times \left(\alpha_2 S(m_1, m_2) - (b_2 - c_2) m_2 + l_2 - \alpha_2 S(m_1^\star(0), 0)\right)^{1-\beta_2}. \end{aligned} \tag{4}$$

Definition 1 is the standard definition of *Nash-in-Nash solution*: essentially, the bargaining outcome in one table is the generalized Nash bargaining solution given the bargaining outcome in the other table.

[8]Note that multiple Nash equilibria may exist in the supply stage, contingent on the pair of compensations (c_1, c_2). Hence it is necessary to include the number of procured slots in the contract space when defining bargaining problem.

Definition 1. *$(c_1^B, m_1^B, l_1^B, c_2^B, m_2^B, l_2^B)$ is a Nash-in-Nash solution in the bargaining stage if and only if*

- *(c_1^B, m_1^B, l_1^B) is the solution of the problem* (3) *given (c_2^B, m_2^B, l_2^B)*
- *(c_2^B, m_2^B, l_2^B) is the solution of the problem* (4) *given (c_1^B, m_1^B, l_1^B).*

3 Analysis

In this section, we analyze the equilibrium of the game outlined in the preceding section. Our first finding is the uniqueness of the equilibrium of the supply stage. This uniqueness is a consequence of the specific characteristics of $S(m_1, m_2)$, notably its *M-concavity*. Our second result reveals that procurers are unable to obtain positive compensation in the bargaining stage. Importantly, these findings are not confined to the scenario with just two hospitals; hence, we extend these results to scenarios involving a general number of hospitals.

3.1 Supply stage

First, we delineate the increment in $S(m_1, m_2)$ with respect to m_1 and m_2. To understand this, we consider the change in the optimal value in the problem defined in (1).

Lemma 1. *Let us denote the optimal matching when the capacities are $(m_1 - 1, m_2)$ as $\mu^{\star,-1}$. The optimal value under (m_1, m_2) is achieved by a matching $\mu^\star$, which is defined as follows: $\mu^\star_{1r'} = \mu^{\star,-1}_{1r'} + 1$ for exactly one r' that maximizes the social surplus with hospital 1 among the unmatched in $\mu^{\star,-1}$, i.e., $r' = \arg\max_{\tilde{r}}\{\Phi_{1\tilde{r}} \mid \tilde{r} \in \mu_0^{\star,-1}\}$, and $\mu^\star_{2r} = \mu^{\star,-1}_{2r}$.*

Proof. It is possible that multiple optimal matchings solve (1). We select one such matching for capacities $(m_1 - 1, m_2)$, denoted by $\mu^{\star,-1}$. When hospital 1 increases its capacity by one, the newly matched refugee with hospital 1 will be either from $\mu_0^{\star,-1}$ or $\mu_2^{\star,-1}$. To maximize social welfare, the candidate r from $\mu_0^{\star,-1}$ should yield the highest social surplus with 1 among $\mu_0^{\star,-1}$, and similarly, the candidate r' from $\mu_2^{\star,-1}$ should do the same among $\mu_2^{\star,-1}$. Suppose in the new matching, r pairs with 1 and the vacant slot in 1 is filled by r'. By the maximization problem, we have:

$$\Phi_{1r} - \Phi_{2r} + \Phi_{2r'} > \Phi_{1r'} \;\Leftrightarrow\; \Phi_{2r'} - \Phi_{1r'} > \Phi_{2r} - \Phi_{1r}. \tag{5}$$

From the dual problem, when denoting the Lagrange multipliers as λ_1^- and λ_2^-, for r we have:

$$\Phi_{2r} - \lambda_2^- \geq \Phi_{1r} - \lambda_1^-, \tag{6}$$

and for r', we obtain:

$$\begin{cases} \Phi_{1r'} - \lambda_1^- \leq 0 \\ \Phi_{2r'} - \lambda_2^- \leq 0. \end{cases} \tag{7}$$

By (6) and (7), we conclude:

$$\Phi_{2r} - \Phi_{1r} \geq \lambda_2^- - \lambda_1^- \geq \Phi_{2r'} - \lambda_1^-.$$

As demonstrated by Shapley and Shubik (1971), multiple sets of Lagrange multipliers can solve the problem. We can select $\lambda_1^- = \Phi_{1r'}$. Hence, $\Phi_{2r} - \Phi_{1r} \geq \Phi_{2r'} - \Phi_{1r'}$. This contradicts (5). Thus, in the new matching, the additional pairing with hospital 1 is with refugee $r' = \arg\max_{\tilde{r}}\{\Phi_{1\tilde{r}} \mid \tilde{r} \in \mu_0^{\star,-1}\}$. □

To further elucidate the problem detailed in (1), we introduce the concept of the largest and smallest Lagrange multipliers.

Definition 2. *Let $\lambda_h^+(m_1, m_2)$ and $\lambda_h^-(m_1, m_2)$ represent the largest and smallest Lagrange multipliers that solve (1) for given capacities (m_1, m_2).*

With these definitions, we describe how the increment in S relates to the increase in capacity through these Lagrange multipliers.

Proposition 1. *The difference $S(m_1, m_2) - S(m_1 - 1, m_2)$ is equal to $\lambda_1^-(m_1 - 1, m_2)$ and also to $\lambda_1^+(m_1, m_2)$. Similarly, the difference $S(m_1, m_2) - S(m_1, m_2 - 1)$ is equal to $\lambda_2^-(m_1, m_2 - 1)$ and $\lambda_2^+(m_1, m_2)$.*

Proof. Focusing on hospital 1, it is evident from the proof of Lemma 1 that $S(m_1, m_2) - S(m_1 - 1, m_2) = \lambda_1^-(m_1 - 1, m_2)$. For capacity (m_1, m_2), the largest Lagrange multiplier equates to the smallest social surplus generated by matched pairs in hospital 1. As shown in the proof of Lemma 1, this is identical to $\lambda_1^-(m_1 - 1, m_2)$. □

The following Corollary 1 then demonstrates that this increment is non-increasing with respect to both m_1 and m_2. This follows logically from the fact that (1) the increment equals the social surplus generated by the new match, and (2) in this transfer utility matching model, matches are formed according to the order of social surplus values.

Corollary 1. *The difference* $S(m_1, m_2) - S(m_1 - 1, m_2)$ *is non-increasing in* m_1 *for all* m_2*, and* $S(m_1, m_2) - S(m_1, m_2 - 1)$ *is non-increasing in* m_2 *for all* m_1*. Moreover, when capacity constraints are not binding, this increment equals* 0*.*

To further our analysis of the equilibrium in the supply stage, we characterize it as a *potential game*, which is discussed in works by Monderer and Shapley (1996); Mavronicolas et al. (2007); Milchtaich (2009).

Proposition 2. *Given a set of parameters,* (b_1, c_1, α_1) *and* (b_2, c_2, α_2)*, we define a function* Ψ *such that* $\Psi(m_1, m_2) \equiv S(m_1, m_2) - \frac{b_1 - c_1}{\alpha_1} m_1 - \frac{b_2 - c_2}{\alpha_2} m_2$*. Then,* Ψ *serves as a weighted potential function.*

Proof. This proposition is demonstrated directly from the definition of π_1 in (2). For any $m_1 \in \mathbb{Z}_+$ and $m_2 \in \mathbb{Z}_+$, and for any $l \leq m_1$,

$$\begin{aligned}\Psi(m_1, m_2) - \Psi(m_1 - l, m_2) &= \frac{1}{\alpha_1}\left(S(m_1, m_2) - S(m_1 - l, m_2) - (b_1 - c_1) l_1\right) \\ &= \frac{1}{\alpha_1}\left(\pi_1(m_1, m_2) - \pi_1(m_1 - l, m_2)\right).\end{aligned}$$

The same relationship is valid for hospital 2. □

Furthermore, we demonstrate that the game possesses a unique pure strategy Nash equilibrium. Initially, it is important to note that all maximizers of the function Ψ correspond to Nash equilibria, as indicated in the work by Mavronicolas et al. (2007). Consequently, the uniqueness of the equilibrium primarily stems from the fact that the potential function Ψ has a unique maximizer. While we leave the proof in Appendix A, we utilize the property called *M-concavity* of Φ to argue the uniqueness as in the discrete convex literature (Murota, 1998).

Theorem 1. *For any given set of parameters, the function* Ψ *has a unique maximizer within the domain* $(m_1, m_2) \in \mathbb{Z}_+^2$*.*

Proof. See Appendix A □

As a direct implication of Theorem 1, we have a unique pure strategy Nash equilibrium in the supply stage game. In light of this, we will henceforth denote the Nash equilibrium in the supply stage, contingent upon the two compensations, as $(m_1^E(c_1, c_2), m_2^E(c_1, c_2))$.

Corollary 2. *In the supply stage of the game, there exists a unique pure strategy Nash equilibrium.*

We provide comparative statics about the equilibrium provision of evacuation slots, particularly focusing on changes in the level of compensation. Adjusting the compensation is seen as a straightforward approach to encourage greater provision. However, it's important to note that increasing compensation for one hospital can potentially hinder provision from the other hospital. Proposition 3 suggests that the path of equilibrium achieved by elevating c_1 while keeping c_2 constant shifts from the upper left to the lower right in the two-dimensional plane defined by m_1 and m_2.

Proposition 3. *The equilibrium number of slots $m_1^E(c_1, c_2)$ is non-decreasing in c_1 and non-increasing in c_2. Conversely, $m_2^E(c_1, c_2)$ behaves in the opposite manner.*

Proof. See Appendix A. □

For clarity, we provide a numerical example in Appendix B to illustrate the theoretical results discussed in this section.

3.2 Bargaining stage

First, we examine the feasible set of utility divisions in the bargaining stage, primarily focusing on the negotiation of hospital 1, though the same principles apply to hospital 2. The bilateral surplus, which is the sum of the utilities of hospital 1 and its procurer in their bargaining, is calculated as follows (omitting the upper script B for simplicity):

$$BS(m_1, m_2) \equiv (1 + \alpha_1)S(m_1, m_2) - b_1 m_1$$

It is important to note that this bilateral surplus is dependent only on the number of procured slots, m_1, given m_2. Let $m_1^{bs}(m_2) = \arg\max_{m_1} BS(m_1, m_2)$ and denote the maximum value by $BS^\star(m_2)$. We posit that the solution to this bargaining game must maximize this bilateral surplus.

Proposition 4. *The equilibrium procurement for hospital 1, m_1^B, equals $m_1^{bs}(m_2^B)$. The same principle applies to hospital 2: $m_2^B = m_2^{bs}(m_1^B)$.*

Proof. Suppose $m_1^B \neq m_1^{bs}(m_2^B)$. In this case, $BS(m_1^{bs}; m_2^B) > BS(m_1^B; m_2^B)$. By shifting to m_1^{bs} and adjusting the lump-sum transfer l_1^B to compensate for any utility loss, both parties in the bargaining can achieve a better outcome. □

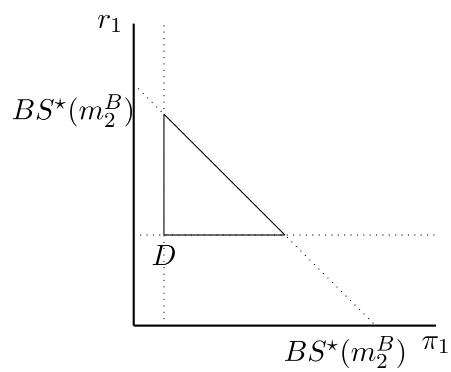

Figure 1. Feasible region of the bargaining stage between hospital 1 and procurer 1. D is the disagreement point. The diagonal line corresponds to the set of the bilateral surplus maximizers.

As a result, the feasible set of utilities can be represented as a triangle, as shown in Figure 1. This convexity of the feasible set indicates that the Nash bargaining problem in this stage is effectively solved by maximizing the Nash product. The figure illustrates the feasible region of the bargaining stage between hospital 1 and procurer 1, where D represents the disagreement point, and the diagonal line corresponds to the set of bilateral surplus maximizers.

3.2.1 Nash-in-Nash Solution

We introduce an assumption regarding the parameter values, particularly focusing on the hospitals' desire to contribute to society, denoted by α_h. We posit that α_h is relatively low, reflecting a scenario where only a small number of evacuation slots are prepared in areas[9]. Specifically, we assume that α_h is sufficiently low to the extent that the payoff from accepting a refugee is lower than the potential benefit of keeping an evacuation slot unused.

Assumption 1. *For all $h \in \{1, 2\}$ and $r \in R$, it holds that $\Phi_{hr} < b_h - \alpha_h \Phi_{hr}$.*

Under Assumption 1, through straightforward algebra, we can establish that $BS(m_1, m_2)$ exhibits a monotonic property. This is particularly useful in characterizing the Nash-in-Nash solution in this stage of bargaining.

Proposition 5. *Given Assumption 1, $BS(m_1, m_2)$ is non-increasing as m_1 increases and non-decreasing as m_2 increases.*

We delve into the analysis of the Nash-in-Nash solution under Assumption 1. A crucial point is that the uniqueness result in the supply stage enables us to simplify the contract space to include

[9] Although our model is static, a low α_h might be considered as a consequence of the low probability of devastating disasters.

only the compensation and lump-sum payoff. This simplification arises because a contract solely based on c_1 sufficiently determines the equilibrium outcome in the supply stage, given the knowledge of c_2, and the same logic applies to hospital 2 when the equilibrium in the lower stage is unique.

Consequently, the Nash bargaining problem between hospital 1 and its procurer, originally detailed in (3), is transformed into the following formulation:

$$c_1^B, l_1^B = \arg\max_{c_1, l_1} \Big(S(m_1^E(c_1, c_2^B), m_2^E(c_1, c_2^B)) - c_1 m_1^E(c_1, c_2^B) - l_1 - S(0, m_2^\star(0)) \Big)^{\beta_1} \times \Big(\alpha_1 S(m_1^E(c_1, c_2^B), m_2^E(c_1, c_2^B)) - (b_1 - c_1) m_1^E(c_1, c_2^B) + l_1 - \alpha_1 S(0, m_2^\star(0)) \Big)^{1-\beta_1}, \tag{8}$$

Similarly, the bargaining between hospital 2 and its procurer, initially specified in (4), is redefined as:

$$c_2^B, l_2^B = \arg\max_{c_2, l_2} \Big(S(m_1^E(c_1^B, c_2), m_2^E(c_1^B, c_2)) - c_2 m_2^E(c_1^B, c_2) - l_2 - S(m_1^\star(0), 0) \Big)^{\beta_2} \times \Big(\alpha_2 S(m_1^E(c_1^B, c_2), m_2^E(c_1^B, c_2)) - (b_2 - c_2) m_2^E(c_1^B, c_2) + l_2 - \alpha_2 S(m_1^\star(0), 0) \Big)^{1-\beta_2}. \tag{9}$$

Focusing on the problem as formulated in (8), we consider the best response given a certain c_2^B. Based on Proposition 4, we understand that the solution must maximize the bilateral surplus. Following Proposition 5, we know that this bilateral surplus reaches its maximum at the smallest m_1 and the largest m_2 within the set of equilibria derived from varying c_1. Consequently, the optimal strategy, for any value of c_2^B, is to select the lowest feasible c_1, as this results in a decrease in $m_1^E(c_1, c_2^B)$ and an increase in $m_2^E(c_1, c_2^B)$. This is due to the fact that, according to Proposition 3, the equilibrium $m_1^E(c_1, c_2^B)$ decreases and $m_2^E(c_1, c_2^B)$ increases as c_1 decreases. The same reasoning applies to the bargaining between hospital 2 and procurer 2.

Therefore, we arrive at the following characterization of the Nash-in-Nash solution for the bargaining stage: there is no monetary compensation for the hospitals, meaning that the number of procured evacuation slots is minimized[10].

Theorem 2. *Under Assumption 1, there exists a unique Nash-in-Nash solution in the bargaining stage, characterized by* $c_1^B = c_2^B = 0$.

[10]This aligns with the real-world situation observed in the procurement of evacuation slots for nuclear incident. For example, in the case of Hamaoka nuclear plants in Japan, local government agents request a certain level of provision without offering any monetary incentives.

We now turn our attention to the equilibrium number of procured evacuation slots. In the scenario where $c_1 = c_2 = 0$, the increase in the function $\Psi(m_1, m_2)$ from the point $(0, 0)$ to $(1, 0)$ is bounded above by $\max\{\Phi_{hr}\} - \frac{b_1}{\alpha_1}$. Given Assumption 1, we have $\max\{\Phi_{hr}\} - \frac{b_1}{\alpha_1} < 0$. This indicates that there is no incentive to deviate from $(0, 0)$ to $(1, 0)$; in other words, the potential function Ψ is uniquely maximized at $(0, 0)$. Consequently, we can anticipate the number of procured evacuation slots in this worst-case scenario to be none.

Theorem 3. *Under Assumption 1, the unique pure strategy Nash equilibrium in the supply stage is* $(m_1, m_2) = (0, 0)$.

Finally, it is important to note that these results are not limited to a market with only two hospitals. The arguments and proofs presented are equally applicable regardless of the number of hospitals in the market.

Theorem 4. *Regardless of the number of hospitals, the supply stage is a potential game where the potential function is M-concave and possesses a unique maximum. For the bargaining stage, under an assumption analogous to Assumption 1, a unique Nash-in-Nash solution exists. In this equilibrium, the level of compensations for all hospitals is zero, and correspondingly, the number of procured evacuation slots for all hospitals is also zero.*

This impossibility result is not surprising, as under Assumption 1, it is less beneficial to allocate space for evacuation slots than to use it for other purposes. Our results above formally confirm this intuition. We consider this scenario because it represents the worst-case situation from the government's perspective when attempting to procure evacuation slots. In the next section, we demonstrate that, even in this worst-case scenario, the government can secure a positive number of evacuation slots in equilibrium by committing to an assignment rule.

4 Rule-Induced Supply

In this section, we explore strategies for successful procurement of evacuation slots. We focus on the challenge faced by the government in maximizing the number of procured slots by altering the assignment rule. Theorem 3 has established that under Assumption 1, if the government's assignment of refugees to hospitals aims solely at maximizing the social surplus, there will be no procurement of evacuation slots. Given this context, our question is whether the government can still procure evacuation slots under such an adverse situation. In other words, we investigate the potential for the government to secure these slots despite the unfavorable circumstances dictated by the assumption that hospitals have a minimal desire to contribute to societal welfare.

4.1 Problem Formulation

We focus on a set of deterministic rules in the sense that $\mathcal{G}$ is a map from R to H_0. This deterministic nature reflects the government's accountability: post-assignment, it is imperative that the government can rationally explain the adopted rule. The problem of the government is formulated in the following way: remember that $m_h^\star(m_{h'};\Gamma)$ is the best response correspondence of hospital h when hospital h' provides $m_{h'}$ under the assignment rule Γ and we include Γ in the expression of S to emphasize the dependence,

$$\left[\begin{array}{ll} \max_{\Gamma\in\mathcal{G}} & m_1^B + m_2^B \\ \text{s.t.} & \begin{aligned} c_1^B, m_1^B, l_1^B = & \mathop{\arg\max}_{c_1, m_1 \in m_1^\star(m_2^B;\Gamma), l_1} \Big(S(m_1, m_2^B;\Gamma) - c_1 m_1 - l_1 - S(0, m_2^\star(0);\Gamma)\Big)^{\beta_1} \\ & \times \Big(\alpha_1 S(m_1, m_2^B;\Gamma) - (b_1 - c_1)m_1 + l_1 - \alpha_1 S(0, m_2^\star(0);\Gamma)\Big)^{1-\beta_1}, \\ c_2^B, m_2^B, l_2^B = & \mathop{\arg\max}_{c_2, m_2 \in m_2^\star(m_1^B;\Gamma), l_2} \Big(S(m_1^B, m_2;\Gamma) - c_2 m_2 - l_2 - S(m_1^\star(0), 0;\Gamma)\Big)^{\beta_2} \\ & \times \Big(\alpha_2 S(m_1^B, m_2;\Gamma) - (b_2 - c_2)m_2 + l_2 - \alpha_2 S(m_1^\star(0), 0;\Gamma)\Big)^{1-\beta_2}. \end{aligned} \end{array}\right. \tag{10}$$

In particular, we focus on the following two assignment rules among $\mathcal{G}$:

1. *Maximum assignment rule* (Γ_{max}): It aims to maximize the social surplus generated by the matching outcomes. This corresponds to the case of stable outcome in the assignment stage.

2. *Reverse assignment rule* (Γ_{rev}): This rule seeks to maximize the reverse social surplus, which is the sum of the inverses of the true social surplus, $\frac{1}{\Phi_{hr}}$.

The structure of the problem (10) is contingent on various parameter values, including b, α, and the social surplus Φ_{hr}. Consequently, the optimal assignment rule should be tailored to these parameters. In this paper, rather than directly tackling this complex problem, we argue that the maximum assignment rule is outperformed by the reverse assignment rule under the reasonable parameter setting. Additionally, under the setting, we contend that the reverse assignment rule resolves the problem: i.e. maximizing the number of procured evacuation slots. This result shows the importance of integrating the procurement process and the assignment process to ensure the sufficient resource supply in the matching market.

4.2 Comparison between Γ_{max} and Γ_{rev}

From Theorem 3, we know that the maximum assignment rule always leads to $(m_1^B, m_2^B) = (0, 0)$. Hence, it is enough to check if there is a set of parameters under which the reverse assignment rule gives us the positive number of evacuation slots in the equilibrium. Before stating the condition of the parameters in detail, for easy understanding, we show a numerical example in which reverse assignment rule can induce the positive number of evacuation slots in the equilibrium.

4.2.1 Example

We set specific values for the parameters: $b_1 = b_2 = 10$ and $\alpha_1 = \alpha_2 = 0.8$. To obtain a more distinct understanding of the outcomes, we concentrate on a scenario where the social surplus, Φ_{hr}, can only take one of two values. Specifically, for all hospitals $h \in \{1, 2\}$ and for all refugees $r \in R$, Φ_{hr} is either $\frac{b_h}{2(1+\alpha_h)}$ or $\frac{b_h}{1+\alpha_h}$. We independently draw Φ_{hr} for each pair of h and r from the set $\{\frac{b_h}{2(1+\alpha_h)}, \frac{b_h}{1+\alpha_h}\}$ with equal probability. Additionally, we define the total number of refugees as 100 and set the maximum number of slots that each hospital can provide to 50. Therefore, $M_1 = M_2 = 50$.

Consider the maximum assignment rule. Figure 2 presents three contour plots of the potential functions for varying pairs of compensations, alongside the contour plot of the bilateral surplus from hospital 1's bargaining. With compensations set at $(c_1, c_2) = (7.5, 7.5)$, a certain number of evacuation slots are provided in the supply stage's Nash equilibrium. However, when c_1 is reduced, the maximizer of the potential function shifts to the upper left, as shown in the second left panel. Since the bilateral surplus for hospital 1 also reaches its maximum at this upper left point, a Nash-in-Nash solution of $c_1 = 0$ emerges in the bargaining stage. The situation is similar for hospital 2. Therefore, in the game's equilibrium, both (m_1, m_2) and (c_1, c_2) converge to $(0, 0)$, as illustrated in the second right panel.

When the government adopts the reverse assignment rule, the story changes significantly. Figure 3 displays four corresponding contour plots for this case. The leftmost panel, illustrating the potential function with $c_1, c_2 = (7.5, 7.5)$, indicates that the potential function is not M-concave. There are two local maximizers: $(m_1, m_2) = (0, 0)$ and $(m_1, m_2) = (50, 50)$. By definition, both points qualify as pure strategy Nash equilibria. Consequently, unlike in the maximum assignment rule scenario, the Nash-in-Nash solution in the bargaining stage must include the number of procured evacuation slots in the contract.

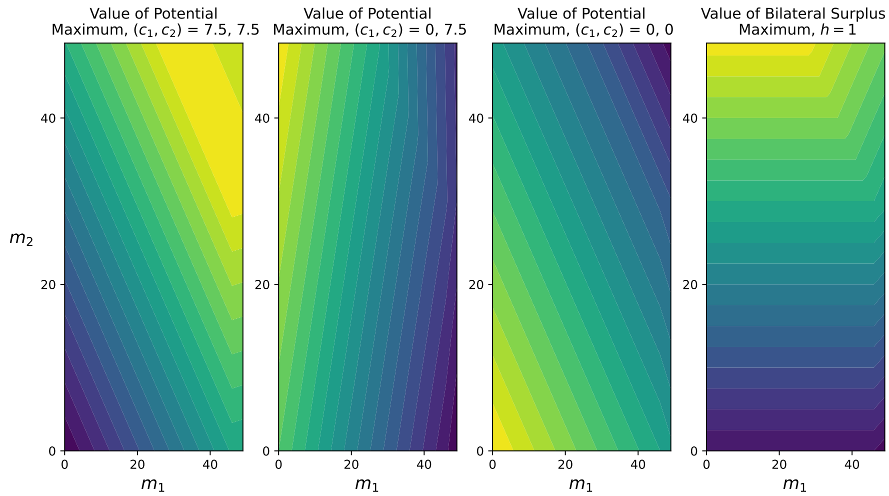

Figure 2. The potential function and the bilateral surplus under the maximum assignment rule. The left three plots are contour plots of the potential function for $(c_1, c_2) = (7.5, 7.5), (0, 7.5), (0, 0)$. The rightmost panel is the contour plot of the bilateral surplus.

Consider a decrease in c_1. Differing from the maximum assignment rule, the second left panel in Figure 3 shows that the point $(0, 50)$ does not emerge as a local maximizer of the potential function. In fact, under some set of assumptions which we describe below, when the government adopts the reverse assignment rule, the set of Nash equilibria in the supply stage is always either $\{(0, 0)\}$ or $\{(0, 0), (50, 50)\}$. Moreover, the bilateral surplus at $(50, 50)$ exceeds that at $(0, 0)$, since $\Phi_{hr} > \frac{b_h}{2(1+\alpha_h)}$. As a result, the Nash-in-Nash solution leads to positive compensations to elicit the highest bilateral surplus among the feasible Nash equilibria in the supply stage. This process results in $(m_1, m_2) = (50, 50)$ being the equilibrium outcome in the game.

4.2.2 Theoretical Property

The first crucial aspect of the aforementioned example is that the bilateral surplus at (M_1, M_2) exceeds 0. This condition ensures that when both (M_1, M_2) and $(0, 0)$ are Nash equilibria, the contract stipulates (M_1, M_2) as the equilibrium to be implemented in the supply stage. This property is secured by imposing a lower bound on the value of Φ_{hr}. Specifically, we adopt the following assumption:

Assumption 2. *For all $r \in R$ and $h \in \{1, 2\}$, $\Phi_{hr} \geq \frac{M_1}{M_1+M_2} \frac{b_1}{1+\alpha_1}$.*

Under Assumption 2, we can confirm that the bilateral surplus at (M_1, M_2) is indeed greater than 0.

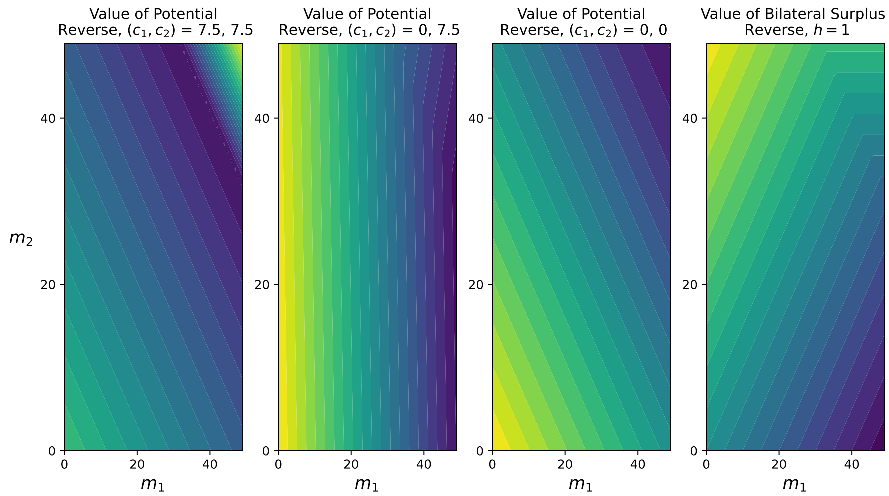

Figure 3. The potential function and the bilateral surplus under the reverse assignment rule. The left three plots are contour plots of the potential function for $(c_1, c_2) = (7.5, 7.5), (0, 7.5), (0, 0)$. The rightmost panel is the contour plot of the bilateral surplus.

Proposition 6. *The bilateral surplus at (M_1, M_2), $BS(M_1, M_2)$, is greater than 0.*

Proof.

$$
\begin{aligned}
(1+\alpha_1)BS(M_1, M_2) - b_1 M_1 &= (1+\alpha_1)\sum_r \Phi_{h(r)r} - b_1 M_1 \\
&> (1+\alpha_1)(M_1 + M_2)\min\{\Phi_{hr}\} - b_1 M_1 \\
&> 0.
\end{aligned}
$$

The final inequality is derived from Assumption 2. □

To achieve clear results, similar to the example, we assume that Φ_{hr} takes either a low value of $\frac{M_1}{M_1+M_2}\frac{b_h}{1+\alpha_h}$ or a high value of $\frac{b_h}{1+\alpha_h}$ for all $h \in \{1, 2\}$ and $r \in R$. It is important to note that this setting adheres to both Assumption 1 and Assumption 2. Additionally, we introduce an assumption about the frequency of these values. Specifically, we posit that for each hospital h, at least M_h refugees are assigned the smaller value, and among all $\{\Phi_{hr}\}_{h,r}$, the higher value occurs at least once.

Assumption 3. *The set $\{\Phi_{hr}\}_{h,r}$ fulfills the following three conditions:*

- *Φ_{hr} is either $\frac{M_1}{M_1+M_2}\frac{b_h}{1+\alpha_h}$ or $\frac{b_h}{1+\alpha_h}$,*

- *For every $h \in \{1, 2\}$, the number of refugees $r \in R$ for which $\Phi_{hr} = \frac{M_1}{M_1+M_2}\frac{b_h}{1+\alpha_h}$ is at least M_h,*

- *There is at least one refugee with a high value for some hospital.*

We mention the validity of Assumption 3 in our context of evacuation of persons with disability. The degree of the disability varies widely and the government classifies them into a finite set of levels[11]. This corresponds to the situation where Φ_{hr} takes the finite number of values. The number of persons with the highest level disability is less than the half of the persons with the lowest level disability[12]. This validates the second point of the assumption.

We outline the characteristics of the Nash-in-Nash solution when the government implements the reverse assignment rule in the assignment stage.

Theorem 5. *Given Assumption 3, when the reverse assignment rule is applied in the assignment stage, there exists a Nash-in-Nash solution where positive compensation is established, and (M_1, M_2) is selected as a Nash equilibrium in the supply stage.*

The essence of this result lies in the complementarity between the provisions of evacuation slots by the hospitals. Under the reverse assignment rule, as the more evacuation slots one hospital procures, an additional match at the other hospital likely benefits the greater surplus generated. This interdependence ensures that the point (M_1, M_2) becomes a Nash equilibrium in the supply stage.

At the same time, the extent of this complementarity is also dependent on the number of slots already filled by refugees. Under Assumption 3, the value of M_h is not high enough for this complementarity to outweigh the cost of increasing the number of procured slots. Consequently, the point $(0, M_h)$ does not emerge as a Nash equilibrium in the supply game. This dynamic highlights how the strategic interaction between the hospitals, shaped by the chosen assignment rule, influences the final outcome in terms of the number of evacuation slots provided.

Proof. See Appendix A.

□

[11] The level is known as "Care need level" which is defined in the Long-Term Care Insurance System of Japan.

[12] Actual number of the persons with the lowest level disability is 1,414,498 and the same of the highest level is 584,917 at 2023. The data is obtained in the reports about the Long-Term Care Insurance System summarized by Ministry of Health, Labour and Welfare: `https://www.mhlw.go.jp/topics/kaigo/osirase/jigyo/m21/2106.html`.

5 Conclusion

In this paper, we examine the resource procurement for a matching market. Our analysis reveals that there is a unique equilibrium where no resources are supplied, which is driven by the substitutability of resources during the bargaining stage.

However, this outcome changes when the government can commit to an assignment rule during the assignment stage. By reversing the order of priorities in the assignment, it becomes possible to secure a positive number of resources in equilibrium by setting an appropriate level of compensation.

In practical situations, such as evacuating persons with disabilities, the government often lacks precise information about hospitals' preferences. Specifically, the importance hospitals place on societal contributions can vary, and the government must address this kind of incomplete information in the procurement process. Exploring a mechanism design approach to this issue presents a promising direction for future research.

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A Proofs

A.1 Proof of Theorem 1

To establish this result, we utilize the concept of *M-concavity* from discrete convex analysis, as discussed in Murota (1998). Consider a function f that takes a set of integers as input, defined as $f : \mathbb{Z}^D \to \mathbb{R}$, where D is the dimension of the inputs. The domain of f, denoted as $domf$, is defined as $\{x \in \mathbb{Z}^D \mid f(x) < \infty\}$.

Definition 3. *(M-concavity) A function f is M-concave if and only if for all $x, y \in domf$ and for any $i \in \{h \mid x_h > y_h\}$, at least one of the following conditions holds, where e_d is the unit vector with its dth element as 1 and all other elements as 0:*

1. *There exists a $j \in \{h \mid x_h < y_h\}$ such that $f(x - e_i + e_j) + f(y + e_i - e_j) \geq f(x) + f(y)$, or*
2. $f(x - e_i) + f(y + e_i) \geq f(x) + f(y)$.

Proof. When Ψ is M-concave, the local maximizer of Ψ is the global maximizer Murota (1998). By Corollary 1, all the nearest points of the local maximizer have the lower values than the value at the local maximizer. Hence, it is enough to show that Ψ is M-concave. When we compare the values of Ψ at two different points, denoted by $(m_1^{'}, m_2^{'})$ and $(m_1^{''}, m_2^{''})$ it is enough the check the following three cases: (i) $m_1^{''} > m_1^{'}$ and $m_2^{''} < m_2^{'}$, and (ii) $m_1^{''} > m_1^{'}$ and $m_2^{''} > m_2^{'}$, and (iii) $m_2^{'} = m_2^{''}$. Because the similar argument is valid for all the cases, we focus on the case of (i) in this proof.

In the case (i), for the condition (2) of M-concavity, Ψ must satisfy the following inequality:

$$\Psi(m_1^{''} - 1, m_2^{''} + 1) + \Psi(m_1^{'} + 1, m_2^{'} - 1) \geq \Psi(m_1^{''}, m_2^{''}) + \Psi(m_1^{'}, m_2^{'}).$$

From the definition of the potential function, this is equivalent to M-concavity of the function S:

$$S(m_1^{''} - 1, m_2^{''} + 1) + S(m_1^{'} + 1, m_2^{'} - 1) \geq S(m_1^{''}, m_2^{''}) + S(m_1^{'}, m_2^{'}).$$

The equivalent condition is

$$S(m_1^{'} + 1, m_2^{'} - 1) - S(m_1^{'}, m_2^{'}) \geq S(m_1^{''}, m_2^{''}) - S(m_1^{''} - 1, m_2^{''} + 1) \tag{11}$$

When we write the change between the two adjunct points as in Figure 4, (11) is equivalent to ② − ① ≥ ⑥ − ⑤. This is what we show below.

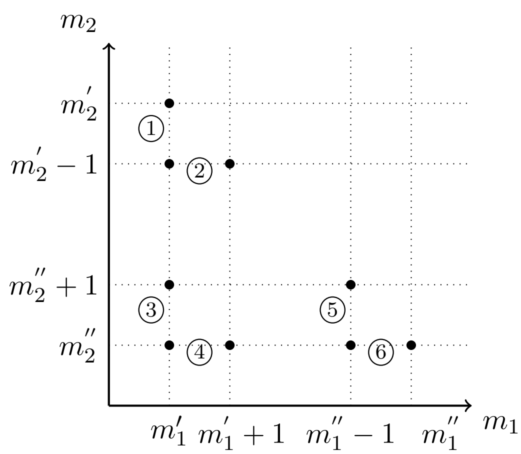

Figure 4. The case of $m_1'' > m_1'$ and $m_2'' < m_2'$.

First, we show the inequality: ② − ① ≥ ④ − ③. Due to Corollary 1, we know that ① ≤ ③. And we also know that ④ ≥ ②. We have the two cases: (1) ④ = ② and (2) ④ > ②. For the first case, we obviously have the inequality. For the second case, we denote the refugee matched with hospital 1 to generate ④ by $r^\star$. $r^\star$ is matched with hospital 2 in the increase of m_2 and so another refugee, denoted by $r^{\star\star}$, is matched with hospital 1 to generate ②. This is why ④ > ②. Because the matching maximizes the social welfare, the welfare gain from replacing $r^{\star\star}$ in $h = 1$ by $r^\star$ in $h = 2$ is less than the welfare gain from replacing $r^\star$ in $h = 2$ by some new refugee:

$$④ - ② \leq \Phi_{2,new} - \Phi_{2,r^\star}$$

Due to Corollary 1, we have ① ≤ $\Phi_{2r^\star}$ and $\Phi_{2,new}$ ≤ ③. Then we have the objective inequality:

$$④ - ② \leq ③ - ① \;\Leftrightarrow\; ② - ① \geq ④ - ③.$$

By symmetry, we have ④ − ③ ≥ ⑥ − ⑤. Then we have ② − ① ≥ ⑥ − ⑤. □

A.2 Proof of Proposition 3

Proof. Let us consider $\hat{c}_1 > c_1$ and denote the corresponding equilibrium number of procured evacuation slots by $\hat{m}^E = (\hat{m}_1^E, \hat{m}_2^E)$ for $(\hat{c}_1, c_2)$ and $m^E = (m_1^E, m_2^E)$ for (c_1, c_2). Assume that $\hat{m}_1^E \neq m_1^E$. Since the equilibrium maximizes the potential function Ψ, the following inequalities are established, where the second inequality stems from $\hat{c}_1 > c_1$:

$$S(\hat{m}) - \frac{b_1 - \hat{c}_1}{\alpha_1}\hat{m}_1^E - \frac{b_2 - c_2}{\alpha_2}\hat{m}_2^E > S(m) - \frac{b_1 - \hat{c}_1}{\alpha_1}m_1^E - \frac{b_2 - c_2}{\alpha_2}m_2^E$$
$$> S(m) - \frac{b_1 - c_1}{\alpha_1}m_1^E - \frac{b_2 - c_2}{\alpha_2}m_2^E > S(\hat{m}) - \frac{b_1 - c_1}{\alpha_1}\hat{m}_1^E - \frac{b_2 - c_2}{\alpha_2}\hat{m}_2^E.$$

By examining the difference between the first and last terms relative to the difference between the second and third terms, we deduce that $(\hat{c}_1 - c_1)(\hat{m}_1^E - m_1^E) > 0$. This implies that $\hat{m}_1^E > m_1^E$ if these two values differ.

Next, we show that when $\hat{m}_1^E > m_1^E$, $\hat{m}_2^E \leq m_2^E$. Remember that the equilibrium level of m_2 is determined by comparing $\frac{b_2-c_2}{\alpha_2}$ with the marginal increment of $S(m_1, m_2)$. Because, for any value of m_2, we have $\lambda_2^+(m_1^E, m_2) \geq \lambda_2^+(\hat{m}_1^E, m_2)$, $\hat{m}_2^E \leq m_2^E$: i.e., the increment is less than $\frac{b_2-c_2}{\alpha_2}$ at the earlier point when $m_1 = \hat{m}_1^E$ than $m_1 = m_1^E$. □

A.3 Proof of Theorem 5

Proof. First we consider the conditions that $(0, M_2)$ is not a Nash equilibrium. The following conditions are sufficient:

$$\begin{cases} \Psi(0, M_2) > \Psi(1, M_2) \ \Leftrightarrow\ \frac{b_1-c_1}{\alpha_1} > S(1, M_2) - S(0, M_2) \\ \Psi(0, M_2 - 1) > \Psi(0, M_2) \ \Leftrightarrow\ \frac{b_2-c_2}{\alpha_2} > S(0, M_2) - S(0, M_2 - 1). \end{cases} \tag{12}$$

Second, we consider the conditions that (M_1, M_2) is a Nash equilibrium. The following conditions are sufficient:

$$\begin{cases} \Psi(M_1, M_2) > \Psi(M_1 - 1, M_2) \ \Leftrightarrow\ S(M_1, M_2) - S(M_1 - 1, M_2) > \frac{b_1-c_1}{\alpha_1} \\ \Psi(M_1, M_2) > \Psi(M_1, M_2 - 1) \ \Leftrightarrow\ S(M_1, M_2) - S(M_1, M_2 - 1) > \frac{b_2-c_2}{\alpha_2}. \end{cases} \tag{13}$$

We focus on hospital 1. Under Assumption 3, the condition for hospital 1 in (12) is assured when the following condition holds:

$$\frac{b_1 - c_1}{\alpha_1} > \frac{M_1}{M_1 + M_2}\frac{b_1}{1 + \alpha_1} \ \Leftrightarrow\ \left(1 + \frac{M_2}{M_1 + M_2}\alpha_1\right)\frac{b_1}{1 + \alpha_1} > c_1.$$

And the condition for hospital 1 in (13) is assured when the following condition holds:

$$\frac{b_1}{1+\alpha_1} > \frac{b_1 - c_1}{\alpha_1} \;\Leftrightarrow\; c_1 > \frac{b_1}{1+\alpha_1}.$$

The pair of the compensations is set to $c_h \in \left[\frac{b_h}{1+\alpha_h}, \left(1 + \frac{M_{-h}}{M_1+M_2}\alpha_h\right)\frac{b_h}{1+\alpha_h}\right]$. Imagine that, while fixing c_2, we decrease c_1 under $\frac{b_h}{1+\alpha_h}$. Then, only $(0,0)$ is Nash equilibrium in the supply stage. Hence, from Proposition 6, to maximize the bilateral surplus, such small c_1 is not agreed in the bargaining stage. When we increase c_1 above $\left(1 + \frac{M_2}{M_1+M_2}\alpha_h\right)\frac{b_h}{1+\alpha_h}$, then the point $(M_1, 0)$ is a new equilibrium. But in the view of bilateral surplus, this point is dominated by (M_1, M_2). Hence this larger c_1 is neither agreed in the bargaining stage. The same argument is applied to hospital 2. Therefore, we know conclude that the pair such that $c_h \in \left[\frac{b_h}{1+\alpha_h}, \left(1 + \frac{M_{-h}}{M_1+M_2}\alpha_h\right)\frac{b_h}{1+\alpha_h}\right]$ is the Nash-in-Nash solution, and the corresponding number of procured evacuation slots is (M_1, M_2). □

B Numerical Example

In order to illustrate the results discussed previously, we present a numerical example involving two hospitals, denoted as $h = 1$ and 2. We consider a scenario with 100 refugees, labeled as $r \in \{1, \cdots, 100\}$. The social surplus generated by a pairing of (h, r), Φ_{hr}, is independently drawn from a standard log-normal distribution. In this example, we ensure that $\Phi_{hr} > 0$, implying that the refugees prefer going to one of the hospitals rather than opting for an outside option.

We initially examine the case of symmetric hospitals: $b_1 = b_2 = 10$, $c_1 = c_2 = 6$, and $\alpha_1 = \alpha_2 = 0.8$. It is important to note that the refugees' preferences over the hospitals are not symmetric. For a specific set of values for Φ_{hr} for each pair of (h, r), the contour plot of the potential function Ψ is illustrated in the left panel of Figure 5. The red marker at $(m_1, m_2) = (4, 7)$ represents the unique maximizer of the potential function. As established in the proof of Theorem 1, this uniqueness arises from Corollary 1, which asserts that S has non-increasing increments. This characteristic is depicted in the right panel of Figure 5, where we observe that with a fixed $m_1 = 4$, the increment of S with respect to the additional capacity of hospital 2 does not increase as m_2 increases.

In order to understand how the equilibrium shifts with varying compensation levels, we conduct an analysis by altering the value of compensation. We fix the parameters as follows: $b_1 = b_2 = 5$, $\alpha_1 = \alpha_2 = 0.8$, and $c_2 = 3.5$. We then compute the equilibrium for different values of c_1, incrementing by 0.5 from 0 to 5. The results are depicted in Figure 6, which plots the equilibrium point corresponding to each value of c_1. In the figure, the darker the color

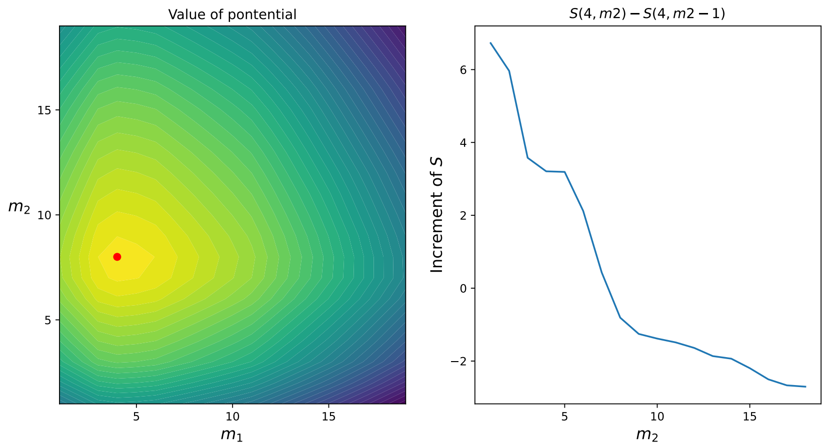

Figure 5. Potential function and non-increasing increment of S w.r.t. m_2 when we fix $m_1 = 4$. The red marker on the left panel is the unique maximizer of the potential function, i.e. the unique pure strategy Nash equilbirium of the supply stage.

of the marker, the higher the value of c_1. Consistent with Proposition 3, we observe that as c_1 increases, m_1 (the number of slots provided by hospital 1) shows a non-decreasing trend, while m_2 (the number of slots provided by hospital 2) exhibits a non-increasing pattern.

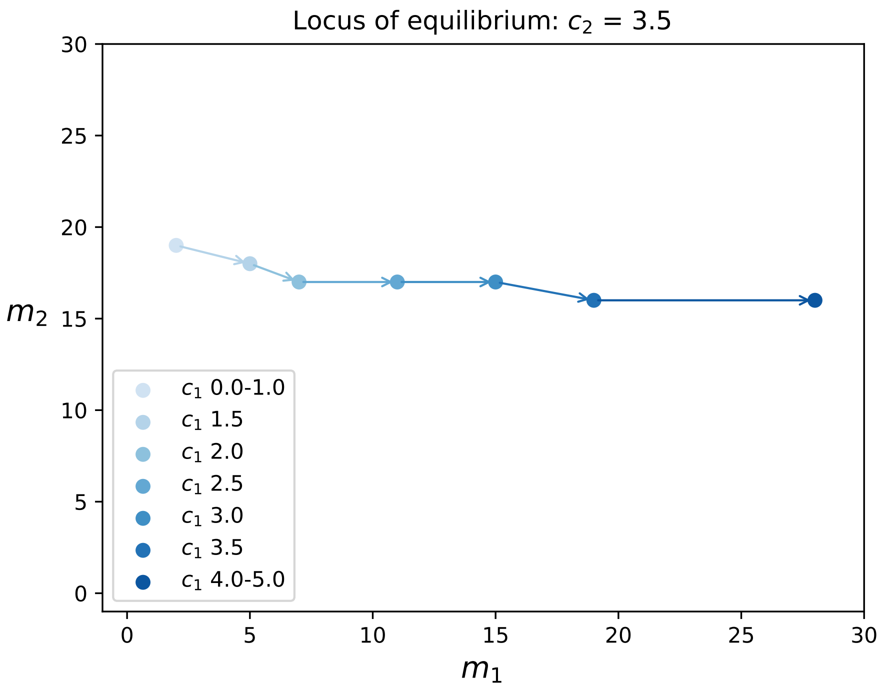

Figure 6. Locus of the equilibrium of the lower stage. As c_1 increases from 0 to 5, the corresponding equilibrium is colored by darker color.